The properties of equilibrium boundary layers

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SUMMARY

The recent work on equilibrium (i.e. self-preserving) turbulent boundary layers in adverse pressure gradients is compared with theoretical predictions based on current generalizations about the turbulent shear flow. Using only assumptions of similarity, it is possible to show that an equilibrium layer can exist only if the free stream velocity varies as a power of distance downstream with an exponent greater than $-\frac{1}{3}$ and if the velocity defect from the free stream is small. Assuming further that the effective eddy viscosity is independent of distance from the wall over the outer part of the layer, most of the properties of equilibrium layers may be computed from the known behaviour of layers in zero pressure gradient. The predicted values of skin friction and the predicted shape and magnitude of the mean velocity distribution are in fair agreement with the observations of Clauser. Finally, the modifications that are necessary if the velocity defect is not small are discussed briefly.

1. INTRODUCTION

Much of the recently published work on turbulent boundary layers has abandoned the long-standing tradition of purely empirical approach based on direct analogy with the behaviour of laminar layers and has begun to use generalizations about the characteristics of turbulent shear flow. The movement began, perhaps, with the work of Ludweig & Tillmann (1949) who established experimentally that the distribution of mean velocity near a smooth wall is independent of the pressure gradient in the free stream, and is determined by the shear stress at the wall and the fluid viscosity. This conclusion, which has since received abundant experimental confirmation (e.g. Coles 1956), is more than an empirical generalization, being a necessary consequence of the mixing-length and similarity theories of turbulence and indeed of any theory of turbulence that assumes net turbulent transfer of energy to be small within the wall region of substantially constant shear stress. At much the same time it was being realized that the apparent variation in shape of the velocity profiles for a constant pressure layer is an illusion, produced by measuring velocities relative to the stationary wall rather than to the free stream. Profiles of velocity defect from the free stream velocity are accurately similar in shape except within the viscous layer next the wall (e.g. Clauser 1954). Again, this 'defect law' is not only an empirical result but is a consequence of two widely accepted generalizations about turbulent flows, that the general motion of a fully developed turbulent flow is independent of the fluid viscosity*, and that developing flows tend to a condition of moving equilibrium or self-preservation.

Although these two results are in a sense experimental rediscoveries of much earlier theoretical work (see, for example v. Kármán (1930) and Goldstein (1938)), they are none the less valuable, and their application to boundary layers in pressure gradients is of great interest. A fundamental advance in this direction was made by Clauser (1954) who, by trial and error adjustment of the external pressure gradient, constructed 'equilibrium' boundary layers satisfying the defect law and who discussed and analysed their behaviour. It is possible, using only the generalizations employed in inferring the existence of a universal velocity profile near a wall and the defect law for constant pressure layers, to determine the conditions necessary for the existence of an equilibrium boundary layer in a pressure gradient (Townsend 1956). The purpose of this paper is to summarize the results of this analysis, including the consequences of assuming an effective eddy viscosity independent of distance from the wall in the outer layer, and then to compare the predictions of this theory with the results and generalizations of Clauser (1956).

2. Notation

Consider a boundary layer on a smooth flat plate whose surface is the plane, $y = 0$. Oz is the direction of homogeneity at right angles to the two-dimensional mean flow, and Ox is parallel to the surface and in the general direction of the flow. Then,

U, V	are the components of the mean velocity parallel to Ox and Oy respectively,
u, v, w	are the components of the turbulent velocity fluctuation parallel to Ox, Oy, Oz,
P	is the pressure,
ν	is the kinematic viscosity,
U_1, P_1	are the Ox component of the mean velocity and the pressure in the free stream just outside the layer,
τ_0	is the shear stress at the wall,
K, A	are the constants in the universal logarithmic distribution of mean velocity near a smooth wall (equation 3.12),
$c_f = 2\tau_0/U_1^2$	is the local coefficient of skin friction,
$\gamma = \tau_0^{1/2}/KU_1$	is a non-dimensional quantity specifying skin friction,
$u_0 = \tau_0^{1/2}/K$	is the velocity scale of the boundary layer,

* This statement is not inconsistent with the dependence of the velocity profile near the wall on fluid viscosity. The magnitude of the viscosity merely determines a velocity of translation for the fully turbulent part of the flow, and velocity differences within it are unaffected.

$\delta = \nu/\tau_0^{1/2}\exp\left(\frac{1}{\gamma}-A\right)$ is the length scale (or 'thickness') of the layer,

η denotes y/δ,

$f(\eta) = (U-U_1)/u_0$ is a universal function specifying the mean velocity distribution,

g_1, g_2, g_{12} are functions specifying the distributions of turbulent intensities and stresses,

$$I_1 = -\int_0^\infty f(\eta)\,d\eta, \qquad I_2 = \int_0^\infty [f(\eta)]^2\,d\eta,$$

$$Y = K^{-3}I_1\gamma^{-2}\exp\left(\frac{1}{\gamma}-A\right), \qquad Y_1 = Y\left\{1+\left(2+\frac{I_2}{I_1}\right)\gamma\right\},$$

$R_x = \int \frac{U_1\,dx}{\nu} = \frac{U_1x/\nu}{1+a}$ is the Reynolds number,

ν_T is the effective eddy viscosity in the outer layer,

$R_s = u_0\,\delta/\nu_T$ is a non-dimensional constant relating the effective eddy viscosity to the velocity distribution,

s is a non-dimensional variable of position,

C is a constant determining the scale of the velocity defect.

It should be noted that the pressures and stresses are 'kinematic', that is, they are the ordinary mechanical values divided by the fluid density. Also, since the pressure in the free stream is related to the velocity by

$$U_1\frac{\partial U_1}{\partial x} = -\frac{\partial P_1}{\partial x}, \tag{2.1}$$

its gradient is specified by the variation of free stream velocity. For the most part, we consider a free stream velocity variation

$$U_1 = Bx^a \quad (x > 0), \tag{2.2}$$

which represents an adverse pressure gradient if the exponent is negative.

3. The conditions for self-preserving development of a boundary layer

The defect law for the mean velocity distribution in a boundary layer is equivalent to the assumption of self-preserving profiles in a wake, i.e.

$$U = U_1+u_0\,f(y/\delta), \tag{3.1}$$

where u_0 and δ are scales of velocity and length depending on x, and the function is independent of x. For the establishment of a self-preserving flow it is necessary that the distributions of shear stress and turbulent intensities should be expressible in terms of the same scales by other functions independent of distance downstream, say

$$\overline{uv} = u_0^2g_{12}(y/\delta), \qquad \overline{u^2} = u_0^2g_1(y/\delta), \qquad \overline{v^2} = u_0^2g_2(y/\delta). \tag{3.2}$$

Naturally, the distributions must satisfy the equations of mean motion, which, for the boundary layer, and to the approximation usually found sufficiently accurate, can be reduced to the single equation,

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+\frac{\partial\overline{uv}}{\partial y}+\frac{\partial(\overline{u^2}-\overline{v^2})}{\partial x} = U_1\frac{\partial U_1}{\partial x}+\nu\frac{\partial^2 U}{\partial y^2}. \tag{3.3}$$

In terms of the self-preserving functions of equations (3.1), and (3.2) this may be written

$$\frac{d(u_0 U_1)}{dx}f-\frac{u_0}{\delta}\frac{d(U_1\delta)}{dx}\eta f'+u_0\frac{du_0}{dx}(f^2+2(g_1-g_2))-$$
$$-\frac{u_0}{\delta}\frac{d(u_0\delta)}{dx}f'\int_0 f\,d\eta-\frac{u_0^2}{\delta}\frac{d\delta}{dx}\eta(g_1'-g_2')+\frac{u_0^2}{\delta}g_{12}' = \frac{\nu u_0}{\delta^2}f'', \tag{3.4}$$

where dashes indicate differentiation with respect to η.

This equation may be satisfied exactly by self-preserving functions independent of x if the ratios of the non-zero coefficients are independent of x. For laminar flow, the functions representing the fluctuating motion are identically zero and self-preserving flow is possible either if

$$u_0 = U_1 \propto x^a, \qquad \delta \propto x^{(1-a)/2}, \tag{3.5}$$

or if

$$u_0 = U_1 \propto e^{\mu x}, \quad \delta \propto e^{-(1/2)\mu x}, \quad (\mu > 0), \tag{3.6}$$

(Goldstein 1939). If δ is now defined by

$$\frac{\delta^2}{\nu}\frac{dU_1}{dx} = \frac{2a}{1+a} = n \quad \text{say},$$

substitution in equation (3.4) gives an equation for $f(\eta)$,

$$anf-f'\left(\eta+\int_0^\eta f\,d\eta\right)+nf^2 = f''. \tag{3.7}$$

(The exponential flow (3.6) corresponds to $n = 2$.) When the non-dimensional stream-function defined by

$$\psi'(\eta) = 1+f(\eta),$$

is introduced, this becomes the Hartree form of the Falkner–Skan equation,

$$\psi'''+\psi\psi'' = n(\psi'^2-1). \tag{3.8}$$

It is known that acceptable solutions satisfying the boundary conditions, $\psi(0)=\psi'(0)=0$, $\psi'''(0)=-n$, $\psi'(\infty)=1$, exist only if $n > -0{\cdot}199$ or $a > -0{\cdot}11$ (Hartree 1937). This sets a lower limit to the possible values of the exponent and to the severity of the adverse pressure gradient.

In turbulent flow, a self-preserving flow is possible only if

$$u_0 \propto U_1 \propto (x_0-x)^{-1}, \qquad \delta \propto (x_0-x), \qquad (x_0 > x), \tag{3.9}$$

corresponding to the accelerated boundary layer within a wedge. The equation may be satisfied *approximately* if the velocity defect is small, that is, if

$$-u_0 f \ll U_1, \tag{3.10}$$

unless $\eta \ll \delta$. Then equation (3.4) may be approximated by

$$\frac{d(u_0 U_1)}{dx}f-\frac{u_0}{\delta}\frac{d(U_1\delta)}{dx}\eta f'+\frac{u_0^2}{\delta}g_{12}' = 0 \tag{3.11}$$

except very close to the wall where direct viscous stresses are appreciable. The existence of a wall layer within which the velocity distribution is given by

$$U = \frac{\tau_0^{1/2}}{K}\left[\log\frac{\tau_0^{1/2}y}{\nu} + A\right] \tag{3.12}$$

implies that we may choose as scales

$$u_0 = \tau_0^{1/2}/K$$

and

$$\delta = \frac{\nu}{\tau_0^{1/2}}\exp\left(\frac{KU_1}{\tau_0^{1/2}} - A\right). \tag{3.13}$$

It may then be shown that self-preserving flow is only possible for free stream velocity variations of three types,

(*a*) $U_1 = Bx^a$ $(x > 0,\ a > -\frac{1}{3})$,
(*b*) $U_1 = B(-x)^a$ $(x > 0,\ a < -\frac{1}{3})$,
(*c*) $U_1 = Be^{\mu x}$ $(\mu > 0)$.

Of these only the first can represent retarded flow (for $0 > a > -\frac{1}{3}$) and correspond with an equilibrium layer of the type studied by Clauser. For this type of velocity variation, the relation between wall stress and position is given by

$$Y = K^{-3}I_1\gamma^{-2}\exp\left(\frac{1}{\gamma} - A\right) = \frac{1}{3a+1}\frac{U_1 x}{\nu} \tag{3.14}$$

with the approximation (3.10). Here $\gamma = \tau_0^{1/2}/KU_1$ is a friction parameter, and $I_1 = -\int_0^\infty f(\eta)\,d\eta$ is a constant of the flow and a function of a. It should be emphasized that the wall stress is uniquely related to the velocity in the free stream and so to the position. The existence of the proper variation of free steam velocity (or pressure) alone is not sufficient to ensure self-preserving development; the boundary layer must be matched to the pressure gradient.

Clauser has observed that the parameter

$$\Pi = \frac{\delta^*}{\tau_0}\frac{dP_1}{dx}, \quad \text{where } \delta^* = \int_0^\infty\left(1 - \frac{U}{U_1}\right)dy,$$

is independent of x in an equilibrium layer. In the present notation,

$$\Pi = -\frac{I_1 u_0 \delta}{K^2 u_0^2}\frac{dU_1}{dx} = -\frac{a}{1+3a}, \tag{3.15}$$

using equations (3.13) and (3.14). The constancy of this parameter is a simple consequence of this very general theory of boundary layers with a small velocity defect. It is also the condition that the terms in the equation for the momentum integral should preserve a constant ratio.

4. The velocity distribution in the outer part of an equilibrium layer

Experience with other self-preserving free turbulent flow suggests that an acceptable description of the mean velocity distribution may be obtained by assuming an effective eddy viscosity constant over the outer part of the layer, the mean velocity distribution due to this being joined smoothly to the logarithmic distribution that obtains in the inner layer (Clauser 1956, Townsend 1956). Clauser has done this without assuming the velocity defect to be small. His results can be derived assuming the relation (3.14) between wall stress and position, although this relation is not accurate for finite velocity defect. In the outer layer, the terms of equation (3.4) that involve the turbulent intensities are usually negligible, and then, putting

$$-\overline{uv} = \nu_T \frac{\partial U}{\partial y}, \tag{4.1}$$

where $u_0 \delta/\nu_T = R_s$, a constant depending on a, the equation of mean motion becomes

$$\frac{I_1}{K^2 R_s} f'' - \frac{2a}{1+3a} f + \frac{1+a}{1+3a} f' \left[\eta + \gamma \int_0 f\, d\eta\right] - \frac{a}{1+3a}\gamma f^2 = 0. \tag{4.2}$$

In terms of a new position variable $s = \left(\frac{K^2 R_s}{I_1}\frac{1+a}{1+3a}\right)^{1/2} \eta$, this is

$$\frac{d^2 f}{ds^2} - nf + \frac{df}{ds}\left(s + \gamma \int_0 f\, ds\right) - \tfrac{1}{2} n\gamma f^2 = 0. \tag{4.3}$$

The non-dimensional stream function defined by

$$\psi'(s) = 1 + \gamma f(s)$$

satisfies the equation

$$\psi''' + \psi\psi'' = \tfrac{1}{2} n(\psi'^2 - 1). \tag{4.4}$$

This is the Hartree equation with two differences. The first and obvious one is that the coefficient outside the bracket is half its value in the equation for laminar flow (3.8), and the second is that the boundary condition at $s = 0$ is no longer $\psi'(0) = 0$ but $\psi'(0) = 1 + \gamma f(0)$. This boundary condition is not independent of x, and does not conform to the initial assumption of self-preservation.

For very small velocity defect in the outer layer, the equation for $f(s)$ becomes

$$\frac{d^2 f}{ds^2} + s\frac{df}{ds} - nf = 0. \tag{4.5}$$

The appropriate solution of this equation is a multiple of the function $Hh_n(s)$, defined by

$$Hh_n(s) = \int_0^\infty \frac{t^n}{n!} \exp[-\tfrac{1}{2}(t+s)^2]\, dt. \tag{4.6}$$

The complete variation of mean velocity is obtained by adjusting the multiplying constant of the function describing the variation in the outer layer to give a smooth junction with the velocity distribution in the inner layer where

$$f(\eta) = \log \eta. \tag{4.7}$$

At high Reynolds numbers, part of the region of constant stress is within the region of small velocity defect, so that the mean velocity distribution within this region of overlap must be both self-preserving and identical with the universal logarithmic profile. The conditions for this have been given as equations (3.13), and this process of matching the velocity profile for constant eddy viscosity to the logarithmic profile is no more than a working approximation consistent with these conditions and the assumptions of self-preservation.

5. The hypothesis of large eddy equilibrium

There is a considerable body of evidence showing that the rate of spread of a turbulent flow into the ambient undisturbed fluid depends on the presence of a system of large eddies which are also responsible for the observed intermittency of turbulent flow near the free stream boundary. The author has postulated that these large eddies are an integral part of the mechanism that regulates the intensity of the turbulent motion and the rate of entrainment of non-turbulent fluid (Townsend 1951, 1956). By assuming a plausible form for these eddies, it is possible to obtain a complete solution for the boundary layer problem in terms of the universal constants, K and A, of equation (3.12) and of two numbers specifying the position and scale of the large eddies with respect to the mean velocity distribution. Estimates of these numbers may be made from a consideration of the origin and dynamics of the large eddies, or they may be determined by analysing experimental measurements of the layers. If it is assumed that these numbers are independent of pressure gradient, all the properties of equilibrium boundary layers can be predicted from the known properties of constant-pressure layers without ambiguity. For details, reference should be made to the full account (Townsend 1956).

To these approximations, the mean velocity distribution consists of the distribution

$$f(\eta) = \log \eta \tag{5.1}$$

in the inner layer (say for $\eta < \eta_0$) and the distribution

$$f(\eta) = -CHh_n(s), \tag{5.2}$$

where

$$s = \left(\frac{K^2 R_s}{I_1(1+n)}\right)^{1/2} \eta, \qquad R_s = \frac{u_0\,\delta}{\nu_T},$$

in the outer layer ($\eta > \eta_0$). As Clauser points out, the experimental determination of mean velocity profiles is not sufficiently accurate to make profitable a comparison of the *shapes* of observed and predicted profiles, and the test of a theory must be the prediction of the scales of variation.

If the velocity distribution in the outer layer is extrapolated to the wall, the extrapolated velocity there, U_0, should be given by

$$K\frac{U_1 - U_0}{\tau_0^{1/2}} = CHh_n(0) = C\frac{\surd(\frac{1}{2}\pi)}{2^{n/2}(\frac{1}{2}n)\,!}. \tag{5.3}$$

The scale of velocity variation is so defined by the constant C, which is given by

$$C^2 = \frac{1}{1+n}\left[\frac{22}{\surd\pi}\,1{\cdot}7^n\left(\frac{1+n}{2}\right)! - \frac{2^{2+n}}{\pi}\left\{\left(\frac{1+n}{2}\right)!\right\}^2\right], \tag{5.4}$$

the constants, 22 and 1.7, being selected to conform to observations of constant-pressure layers. If $(1+n)$ is small,

$$C^2 = \frac{6\cdot 66}{1+n}. \tag{5.5}$$

Clauser uses two scales of distance from the wall. The first is

$$\Delta = \int_0^\infty \frac{U_1 - U}{\tau_0^{1/2}}\, dy = \frac{I_1}{K}\delta. \tag{5.6}$$

Distances expressed in this scale are related to distances expressed in the scale of the variable s (equation (4.3)) by

$$\frac{y}{\Delta} = \left(\frac{1+n}{I_1 R_s}\right)^{1/2} s = [k(1+n)]^{1/2} s. \tag{5.7}$$

The second scale is δ_0, the total thickness of the layer, presumably the distance from the wall beyond which the mean velocity variation is imperceptible. This scale will be roughly three times the scale of s.

Clauser expresses the eddy viscosity in the outer layer in terms of the non-dimensional constant, k, which is given by

$$\frac{1}{k} = \frac{U_1\delta^*}{\nu_T} = I_1 R_s = \frac{121\, K^{-2} \quad 1\cdot 2^{2n}}{\frac{22}{\sqrt{\pi}}\, 1\cdot 7^n \left(\frac{1+n}{2}\right)! - \frac{2^{2+n}}{\pi}\left\{\left(\frac{n+1}{2}\right)!\right\}^2}. \tag{5.8}$$

The comparison of these predictions with Clauser's measurements is shown in figures 1 and 2, using the values of n in table 1 computed from equation (3.15). This procedure gives a definite value for a, and should

	Π	a	$1+n$	$1+3a$
Distribution I	2	$-0\cdot 286$	$0\cdot 20$	1/7
Distribution II	7	$-0\cdot 318$	$0\cdot 0667$	1/22

Table 1

make some allowance for the effect of a finite velocity defect. These values of a are in fair agreement with the observed distributions of free stream velocity (figure 3).

The computed variation of C agrees fairly well with observation, the difference being most marked for the lesser pressure gradient. The theory also predicts a nearly constant value for k in agreement with Clauser's findings, although the computed values are 20–25 % less than the observed values. This is caused by the use of the velocity profile (5.2) to compute I_1. By overestimating the velocity defect near the free stream, this leads to values of I_1 about 10% greater than the true values. The constancy of k is not unexpected as $U_1\delta^* = \int_0^\infty (U_1 - U)\, dy$ is nearly the simplest possible expression for the product of length and velocity scales for distributions of nearly similar shapes.

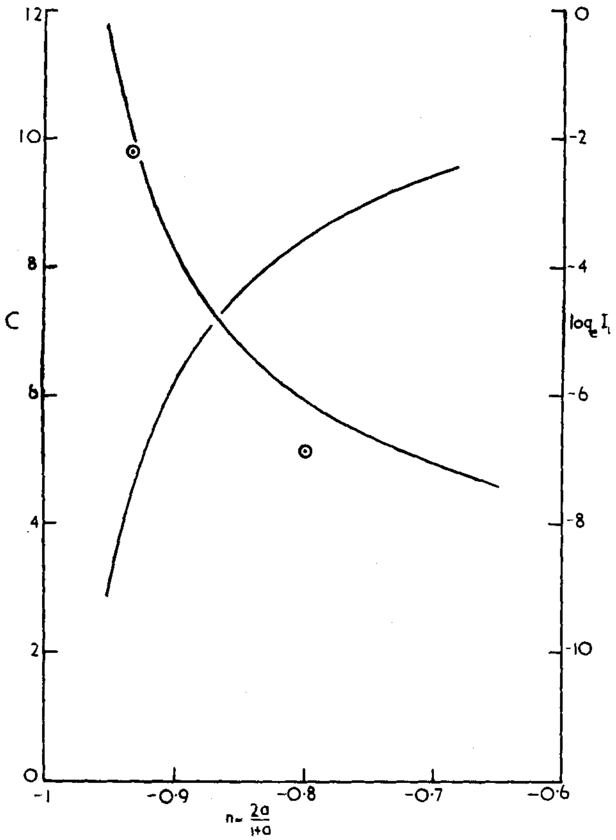

Figure 1. Computed variations of the constants C and I_1 with n (circled points from Clauser's measurements).

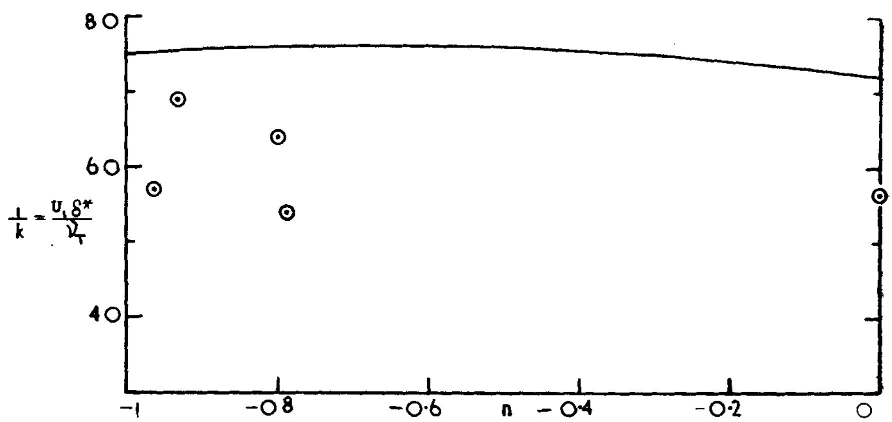

Figure 2. Dependence of effective eddy viscosity on integrated velocity defect (full line from theory, circled points from Clauser 1956).

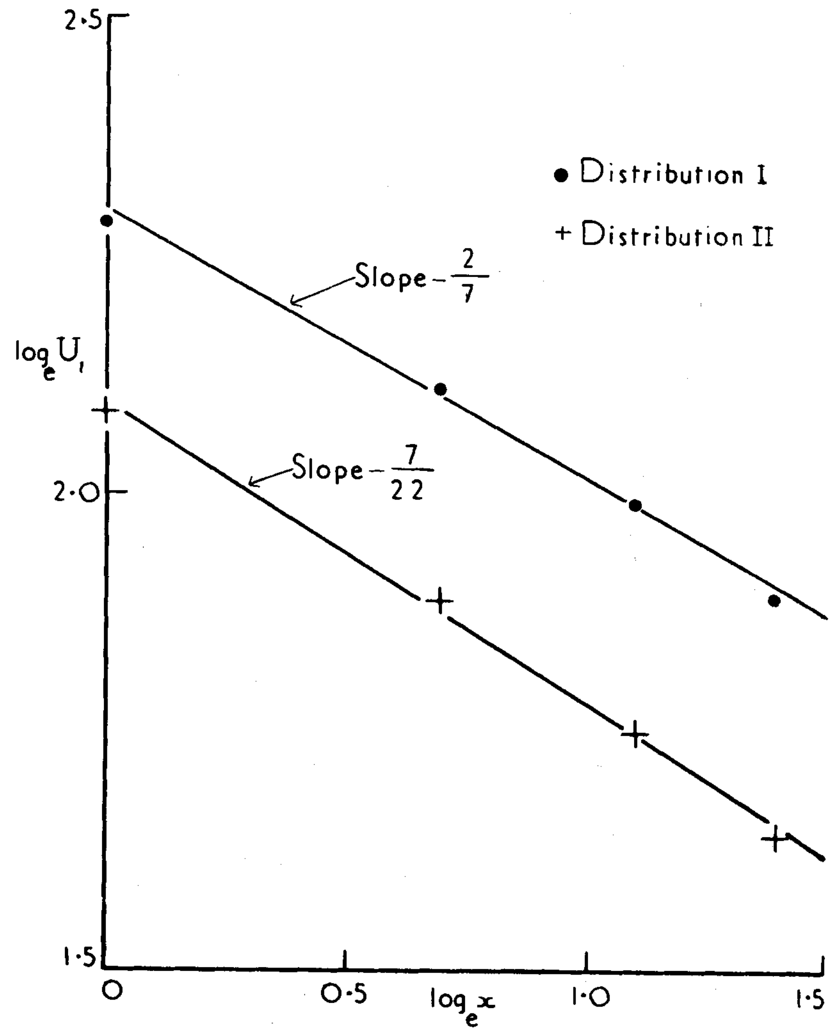

Figure 3. Comparison of Clauser's pressure distributions with the appropriate power laws.

Comparison of predicted and observed scales of distance may be made by computing the position at which the velocity defect is one-half of the extrapolated velocity defect, $U_1 - U_0$, or the position at which the velocity defect is small. Some results of this comparison are shown in table 2.

Lastly, the skin friction may be computed from the friction equation

$$Y = K^{-3} I_1 \gamma^{-2} \exp\left(\frac{1}{\gamma} - A\right) = \frac{1}{1+3a} \frac{U_1 x}{\nu}, \tag{5.9}$$

using values of I_1 given by

$$I_1 = [(1+n)C^2 + 1] \exp\left[1 - \frac{(\frac{1}{2}\pi)^{1/2} C}{2^{n/2} (\frac{1}{2}n)\,!}\right]. \tag{5.10}$$

If $1+n$ is small, this is nearly

$$I_1 = 7{\cdot}66 \exp\left[1 - \left(\frac{6{\cdot}66}{1+n}\right)^{1/2}\right]. \tag{5.11}$$

	Δ/δ_0	$y_{1/2}/\delta_0$	$s_{1/2}$ Obs.	$s_{1/2}$ Calc.	s_0 Obs.	$Hh_n(s_0)$
Distribution I	6·35	0·41	1·26	1·18	3·06	0·010
Distribution II	12·0	0·45	1·26	1·18	2·81	0·018
Constant pressure	3·58	—	—	—	2·37	0·025

Table 2

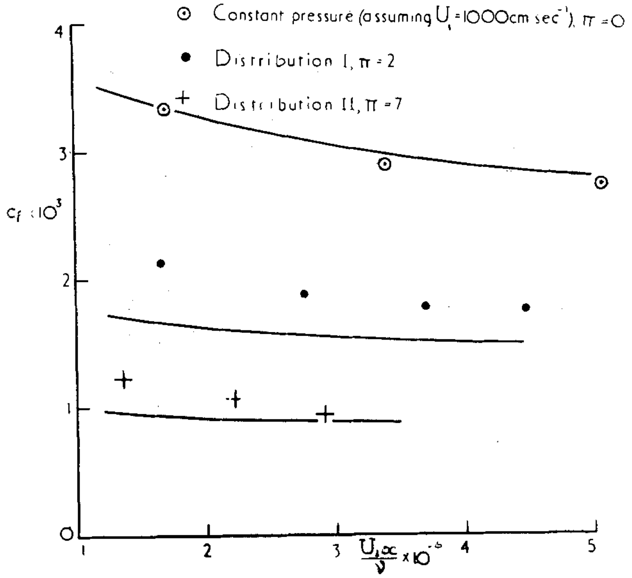

Figure 4. Observed and theoretical distributions of friction coefficient.

Figure 4 compares the predicted values of local friction coefficient with the experimental measurements. The agreement is good only for the larger pressure gradient.

6. The critical value of the power law exponent

The theory for small velocity defect in the outer layer, which is valid at extremely high Reynolds numbers (equation (3.14) shows γ to decrease indefinitely with increasing Reynolds number), predicts that equilibrium layers exist only if $a > -\frac{1}{3}$. The basis of this prediction is the necessity for a positive value of the friction function Y. If the effective eddy viscosity is constant within the outer layer, equation (4.6) shows that self-consistent velocity profiles exist, the critical profile for $a = -\frac{1}{3}$ being

$$f(s) = -Ce^{-1/2s^2}, \tag{6.1}$$

which has a zero gradient at the wall indicating zero wall stress.

There are two simple ways of looking at the consequences of finite velocity defect, viz. by including second-order terms in the derivation of the stress-position relation (assuming self-preservation of velocity profiles and wall stress) or by finding the range of a for which the modified Hartree equation (4.4) has acceptable solutions (assuming that the coefficients of equation (3.4) may be computed from the relation (3.14)). Ideally, both should be investigated but the result is a modified form of equation (4.4) whose solutions have not been studied.

The equation for the momentum integral is

$$\frac{d}{dR_x}\left[\{I_1-(2I_1+I_2)\gamma\}\gamma^{-2}\exp\left(\frac{1}{\gamma}-A\right)\right]+ \\ +\frac{\nu}{U_1^2}\frac{dU}{dx}(2I_1-I_2\gamma)\gamma^{-2}\exp\left(\frac{1}{\gamma}-A\right) = K^3, \quad (6.2)$$

where terms of order less than γ have been omitted. With the use of friction function

$$Y_1 = K^{-3}[I_1-(2I_1+I_2)\gamma]\gamma^{-2}\exp\left(\frac{1}{\gamma}-A\right), \quad (6.3)$$

it becomes

$$\frac{dY_1}{dR_x}+\frac{2a}{1+a}[1+(2+\tfrac{1}{2}I_2/I_1)\gamma]\frac{Y_1}{R_x} = 1 \quad (6.4)$$

to the same approximation. Since γ is a slowly varying quantity, an approximate solution may be obtained by neglecting its variation. This is

$$Y_1 = R_x\frac{1+a}{1+3a+2a\gamma(2+\frac{1}{2}I_2/I_1)}, \quad (6.5)$$

which has meaning only if

$$a > -\frac{1}{3+2a\gamma(2+\frac{1}{2}I_2/I_1)}. \quad (6.6)$$

The quantity $\gamma I_2/I_1$ is nearly proportional to the velocity defect ratio $(U_1-U_0)/U_1$. For the critical velocity profile defined by equation (6.1),

$$\frac{I_2}{I_1}\gamma = \frac{C}{\sqrt{2}}\gamma = \frac{1}{\sqrt{2}}\frac{U_1-U_0}{U_1}. \quad (6.7)$$

Alternatively, we may suppose the development of the layer to be described by the theory for small velocity defect, and look at the conditions for acceptable solutions of the modified Hartree equation (4.4) for various defect ratios. Numerical solutions quoted by Clauser (1956) show that the critical value of the exponent varies with defect ratio in the way shown in figure 5. A comparison with critical values obtained from equations (6.6) and (6.7) shows fair agreement between the two estimates only for small velocity defect ratios.

Neither of these arguments is altogether satisfactory and further work is desirable.

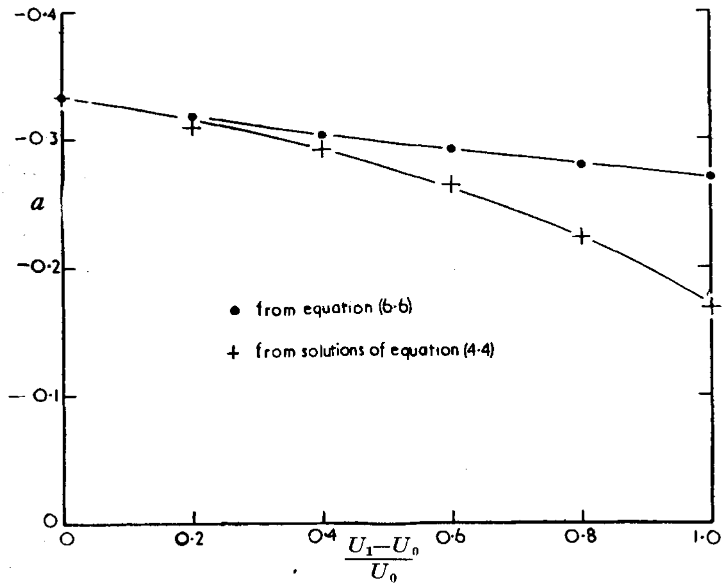

Figure 5. Critical values of the power-law exponent.

7. Concluding remarks

The properties of self-preserving or equilibrium boundary layers may well prove to be of considerable importance in the understanding of the general behaviour of boundary layers in pressure gradients. The theory based on the assumptions of small velocity defect and large eddy equilibrium fits the observations of Clauser as well as can be expected in view of the comparative crudity of the theory and the absence of disposable constants. It should be noted that the hypothesis of large eddy equilibrium is very nearly equivalent to assuming eddy viscosity to be a universal multiple of the integral of the velocity defect.

The condition that the velocity defect is small is satisfied very weakly in the experimental layers, and better agreement between theory and experiment might be obtained by considering the effects of finite defect.

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